Honors Pre-calculus: Vector and Parametric Equation Practice Problems

1. A particle moves along a line in a plane (rectilinear motion). At $t = 0$, $s = (3, -1)$; at $t = 2$, $s = (5, 5)$.

(a) Represent the movement of particle geometrically as vector $\overrightarrow{AB}$

(b) Represent $\overrightarrow{AB}$ in component form $\left( \begin{array}{c} 2 \\ 6 \end{array} \right)$

(c) The magnitude of $\overrightarrow{AB}$ ($|\overrightarrow{AB}|$) is: $\sqrt{4+0} = 2\sqrt{10}$

(d) The slope of $\overrightarrow{AB}$ is: $3$

(e) The velocity of the particle as vector $\overrightarrow{v}$ is: $\left( \begin{array}{c} 1 \\ 3 \end{array} \right)$

(f) The average speed of the particle is: $\sqrt{10}$

(g) Write a vector equation of the line that contains $\overrightarrow{AB}$

\[
(x, y) = (x_0, y_0) + t(\overrightarrow{v})
\]

\[
\left( \begin{array}{c} x \\ y \end{array} \right) = \left( \begin{array}{c} 3 \\ -1 \end{array} \right) + t \left( \begin{array}{c} 1 \\ 3 \end{array} \right)
\]

initial position + time parameter * constant velocity

(h) Write parametric equations for the coordinates of the particle at time $t$: $x = 3 + t$, $y = -1 + 3t$

(i) Find a Cartesian equation (rectangular equation in $x$ and $y$) of the line containing $\overrightarrow{AB}$
(i.e. eliminate the parameter $t$ and identify the graph of the parametric equations)

(1) solve for $t$ in the $x$ parametric equation

\[x - 3 = t\]

(2) substitute for $t$ in the $y$ parametric equation

\[y = -1 + 3(x - 3)\]

\[y = 3x - 10\]

(j) When and where does the particle cross the line $2x - y = -3$?
(i.e. what is the intersection of $\overrightarrow{AB}$ and the line $y = 2x + 3$?)

\[2(3 + t) - (-1 + 3t) = -3\]

\[6 + 2t + 1 - 3t = -3\]

\[7 - t = -3\]

\[10 = t\]

\[\cap \{x = 3 + 10, y = -1 + 3(10)\}

\[(13, 29)\]
2. A particle moves along a line in a plane (rectilinear motion). At \( t = 0 \), \( s = (3, 1) \); at \( t = 3 \), \( s = (-6, 4) \).

(a) Represent the movement of particle geometrically as vector \( \overrightarrow{AB} \)

(b) Represent \( \overrightarrow{AB} \) in component form \((-9, 3)\)

(c) The magnitude of \( \overrightarrow{AB} \) \( (|\overrightarrow{AB}|) \) is: \( \sqrt{9^2 + 3^2} = \sqrt{90} = 3\sqrt{10} \)

(d) The slope of \( \overrightarrow{AB} \) is: \( -\frac{3}{9} = -\frac{1}{3} \)

(e) The velocity of the particle as vector \( \vec{v} \) is: \((-3, 1)\)

(f) The average speed of the particle is: \( \sqrt{10} \)

(g) Write a vector equation of the line that contains \( \overrightarrow{AB} \) \((x, y) = (x_0, y_0) + t(\vec{v})\)
   initial position + time parameter * constant velocity

(h) Write parametric equations for the coordinates of the particle at time \( t \): \( x = 3 - 3t \quad y = 1 + t \)

(i) Find a Cartesian equation (rectangular equation in \( x \) and \( y \)) of the line containing \( \overrightarrow{AB} \)
   (i.e. eliminate the parameter \( t \) and identify the graph of the parametric equations)
   1. solve for \( t \) in the \( x \) parametric equation \( t = 1 - \frac{x}{3} \)
      \( 1 - \frac{1}{3}x \)
   2. substitute for \( t \) in the \( y \) parametric equation \( y = 1 + (1 - \frac{1}{3}x) = 2 - \frac{1}{3}x \)

(j) When and where does the particle cross \( x^2 + y^2 = 4 \)?
   (i.e. what is the intersection of \( \overrightarrow{AB} \) and the circle \( x^2 + y^2 = 4 \))

\[
(3 - 3t)^2 + (1 + t)^2 = 4 \\
9 - 18t + 9t^2 + 1 + 2t + t^2 = 4 \\
10t^2 - 16t + 10 = 4 \\
10t^2 - 16t + 6 = 0 \\
2(5t^2 - 8t + 3) = 0 \\
2(5t - 3)(t - 1) = 0 \\
t = \frac{3}{5}, 1 \\
\begin{align*}
\text{\( t = \frac{3}{5}, 1 \)} & \quad \text{\( \left( \frac{6}{5}, \frac{8}{5} \right) \) and} \\
\end{align*}
\]
\[
\sin^2 t + \cos^2 t = 1
\]

3. Parametric equations of curves

(a) Write each expression in terms of \(\sin t\): 
\[
\begin{align*}
\cos^2 t &= 1 - \sin^2 t \\
\cos 2t &= 1 - 2\sin^2 t \\
\csc t &= \frac{1}{\sin t}
\end{align*}
\]

(b) Show by substitution that the equations \(x = \sin t\) and \(y = 1 - \cos 2t\) are parametric equations of the parabola \(y = 2x^2\)

\[
\begin{align*}
y &= 2x^2 \\
1 - \cos 2t &= 2 \left(\sin^2 t\right) \\
1 - (1 - 2\sin^2 t) &= 2\sin^2 t \\
2\sin^2 t &= 2\sin^2 t \\
\end{align*}
\]

(c) Do the parametric equations have the same graph as the rectangular equation of the parabola? \(\text{No}\)

(Find the domain and range of each equation)

\[
\begin{align*}
x &= \sin t \\
y &= 1 - \cos 2t
\end{align*}
\]

\[
\begin{align*}
D &= [-1, 1] \\
R &= [0, \infty)
\end{align*}
\]

(d) Eliminate the time parameter and identify the graph of the parametric curve: \(x = t^2 - 2, \ y = 3t\)

(Suggestion: solve the \(y\) equation for \(t\) and substitute into the \(x\) equation)

\[
\begin{align*}
\frac{1}{3}y &= t \\
x &= \left(\frac{1}{3}y\right)^2 - 2 \\
x &= \frac{1}{9}y^2 - 2 \\
x + 2 &= \frac{1}{9}y^2 \\
y &= \pm\sqrt{9x + 18}
\end{align*}
\]

(e) Eliminate the time parameter and identify the graph of the parametric curve: \(x = 2\cos t, \ y = 2\sin t\)

(Show by substitution that the parametric equations represent a circle \(x^2 + y^2 = r^2\) for \(0 \leq t \leq 2\pi\))

\[
\begin{align*}
(2\cos t)^2 + (2\sin t)^2 &= r^2 \\
4\cos^2 t + 4\sin^2 t &= r^2 \\
4\left(\cos^2 t + \sin^2 t\right) &= r^2 \\
4(1) &= r^2 \\
x^2 + y^2 &= 4
\end{align*}
\]
4. \( \vec{u} = (2, -3), \ \vec{v} = (-8, 12), \) and \( \vec{w} = (6, 4) \).

(a) \[ |\vec{u}| = \sqrt{2^2 + (-3)^2} = \sqrt{13} \]

(b) What is the dot product of \( \vec{u} \) and \( \vec{v} \) \( (\vec{u} \cdot \vec{v}) \)?

\[ 2 \cdot (-8) + (-3) \cdot 12 = -52 \]

(c) Are any of the three vectors perpendicular? Show your work.

\[
\begin{align*}
\vec{u} \cdot \vec{v} &= -52 \\
\vec{u} \cdot \vec{w} &= 2 \cdot 6 + (-3) \cdot 4 = 0 \quad \vec{u} \perp \vec{w} \\
\vec{v} \cdot \vec{w} &= (-8) \cdot 6 + (12) \cdot 4 = 0 \quad \vec{v} \perp \vec{w} 
\end{align*}
\]

(d) Are any of the three vectors parallel? Show your work.

\[ \vec{u} \parallel \vec{v} \]

\[
\begin{align*}
\frac{\vec{u}}{u} &= \frac{-3}{2} \\
\frac{\vec{v}}{v} &= \frac{-3}{2}
\end{align*}
\]

5. \( \vec{s} = (6, -5), \) and \( \vec{t} = (-7, 4) \). Find the angle between \( \vec{s} \) and \( \vec{t} \).

\[
\cos \theta = \frac{\vec{s} \cdot \vec{t}}{|\vec{s}| \cdot |\vec{t}|} = \frac{(6)(-7) + (-5)(4)}{\sqrt{6^2 + (-5)^2} \cdot \sqrt{(-7)^2 + 4^2}} = \frac{-42 - 20}{\sqrt{61} \cdot \sqrt{65}} = \frac{-62}{\sqrt{61} \cdot \sqrt{65}}
\]

\[ \theta = \cos^{-1} \left( \frac{-62}{\sqrt{61} \cdot \sqrt{65}} \right) \approx 169.939^\circ \]