Pre-calculus – Rational Expressions: Arithmetic operations and simplification

A rational expression is a fraction with numbers and/or variables. The rules for fractions apply.

1. To simplify a rational expression, factor the numerator & denominator and cancel any common factors.

E.g. \( \frac{9a^2b}{18ab^2} = \frac{9ab(a)}{9ab(2b)} = \frac{a}{2b} \)

E.g. \( \frac{2a - 8}{a^2 - a - 12} = \frac{2(a - 4)}{(a - 4)(a + 3)} = \frac{2}{a + 3} \)

E.g. \( \frac{6n}{6 + n} \) cannot be simplified

E.g. \( \frac{x^2 - 25}{x^2 + 10x + 25} = \frac{(x - 5)(x + 5)}{(x + 5)(x + 5)} = \frac{x - 5}{x + 5} \)

E.g. \( \frac{2x - 2y}{y^2 - x^2} = \frac{2(x - y)}{(y - x)(y + x)} = \frac{-1}{2} \frac{(x - y)}{(y - x)(y + x)} = \frac{-1}{2} \frac{(x - y)}{(y + x)} = \frac{-2}{x + y} \)

2. To add or subtract fractions:

(a) if the denominators are the same, add/subtract numerators, and place the sum/difference over the common denominator.

E.g. \( \frac{3a + 2 - a - 3}{a - 7} = \frac{3a + 2 - a + 3}{a - 7} = \frac{2a + 5}{a - 7} \)

E.g. \( \frac{8n + 3 - 2n - 5}{3n + 4} = \frac{6n + 8 - a + 3}{3n + 4} = \frac{2(3n + 4)}{3n + 4} = 2 \)

(b) if the denominators are different, convert both fractions over the lowest common denominator, the LCD, and then add/subtract numerators, and place the sum/difference over the LCD.

E.g. \( \frac{1}{a} + \frac{1}{b} = \frac{1b}{ab} + \frac{1a}{ab} = \frac{b + a}{ab} \) or \( \frac{a + b}{ab} \)

Note: \( \frac{1}{a} + \frac{1}{b} \neq \frac{2}{a + b}; \neq \frac{2}{ab} \)

E.g. \( \frac{6}{5x} + \frac{7}{10x^2} = \frac{6(2x)}{5x(2x)} + \frac{7}{10x^2} = \frac{12x}{10x^2} + \frac{7}{10x^2} = \frac{12x + 7}{10x^2} \)

Sometimes the rational expression can be simplified, after addition or subtraction is completed over the LCD, by canceling common factors.

\( \frac{6}{x^2 - 4x - 5} + \frac{1}{x + 1} = \frac{6}{(x - 5)(x + 1)} + \frac{1(x - 5)}{(x + 1)(x - 5)} = \frac{6 + x - 5}{(x + 1)(x - 5)} = \frac{x + 1}{x - 5} = \frac{1}{x - 5} \)

To add (or subtract) a whole or mixed number and a fraction, first convert the whole or mixed number to an improper fraction over the LCD.

E.g. \( x - 2 + \frac{3}{x + 2} = \frac{x(x + 2)}{x + 2} - \frac{2(x + 2)}{x + 2} + \frac{3}{x + 2} = \frac{x^2 + 2x - 2x - 4 + 3}{x + 2} = \frac{x^2 - 1}{x + 2} \)
II. To multiply fractions: (a) cross-cancel to reduce fractions, then (b) multiply numerators separately, and multiply denominators separately.

To multiply a fraction by an integer, multiply only the numerator by the integer.

E.g. \( \frac{3x}{y} \cdot \frac{6x}{2y} = \frac{6x}{y} \)

\[
\frac{2a^2d \cdot 9b^2c}{3bc \cdot 16cd^2} = \frac{1}{b} \cdot \frac{a^2d}{d^2} \cdot \frac{3}{b} \cdot \frac{b^2c}{y} = \frac{3ab}{yd}
\]

\[
\frac{x + 5}{3x} \cdot \frac{12x^2}{x^2 + 7x + 10} = \frac{x + 5}{3x} \cdot \frac{12x^2}{(x + 5)(x + 2)} = \frac{4x}{x + 2}
\]

\[
\frac{x^2 - x - 6}{9 - x^2} \cdot \frac{x^2 + 7x + 12}{x^2 + 4x + 4} = \frac{(x - 3)(x + 2)}{(3 - x)(x + 3)} \cdot \frac{(x + 4)(x + 3)}{(x + 2)(x + 2)} = \frac{(x - 3)(x + 4)}{(3 - x)(x + 2)}
\]

\[
\frac{(x - 3)(x + 4)}{(3 - x)(x + 2)} \cdot \frac{1}{(-4 - x - 4)} = \frac{-x + 4}{x + 2}
\]

III. To divide fractions:

(a) if the fractions are written horizontally, multiply the first fraction by the reciprocal of the divisor (2nd fraction); i.e. flip the second fraction, then multiply the numerators and multiply the denominators;

Note: never flip the first fraction.

Like multiplication of fractions, cancel common factors, either in a fraction or by cross-cancellation.

Note: only cross-cancel when the fractions are in multiplication mode, never across a division sign.

\[
\frac{3x}{x + 2} \div \frac{3x^2 - 3x}{x^2 - 4} = \frac{3x}{x + 2} \cdot \frac{x^2 - 4}{3x^2 - 3x} = \frac{3x}{x + 2} \cdot \frac{(x - 2)(x + 2)}{3x(2x - 1)} = \frac{x - 2}{x - 1}
\]

\[
\frac{3m^2 + 7m - 6}{2m^2 + 11m + 15} \div \frac{2m^2 - 3m - 20}{6m^2 + 2m - 4} = \frac{3m^2 + 7m - 6}{2m^2 + 11m + 15} \cdot \frac{6m^2 + 2m - 4}{2m^2 - 3m - 20} = \frac{(3m - 2)(m + 3)}{(2m + 5)(m - 4)} \cdot \frac{(2m + 5)(m - 4)}{2(3m - 2)(m + 1)} = \frac{m - 4}{2(m + 1)}
\]

(b) if the fraction is written vertically, multiply the numerator by the reciprocal E.g. \( \frac{3}{4} \cdot \frac{6}{5} = \frac{9}{10} \)

of the denominator; i.e. flip the denominator and multiply by the numerator. E.g. \( \frac{a}{b} = \frac{a}{b} \cdot \frac{1}{c} = \frac{a}{bc} \)
Remember that you can cross-cancel only across a multiplication sign, never across a division sign.

You can reduce a fraction at any time.

To simplify (divide) complex fractions, simplify the numerator and denominator into simple fractions, then multiply the first fraction by the reciprocal of the second fraction.

\[
\frac{\frac{1}{a} \cdot \frac{1}{b}}{\frac{1}{b} + \frac{1}{a} + \frac{1}{b} - \frac{1}{a}} = \frac{\frac{ab}{b^2} - \frac{ab}{a^2}}{\frac{b+a}{ab}} = \frac{\frac{b-a}{a^2} \cdot \frac{a^2}{b^2}}{\frac{b+a}{ab}} = \frac{b-a}{b+a} \text{ or } \frac{b-a}{a+b}
\]

\[
\frac{a-2}{a+3} + \frac{3}{a-3} = \frac{(a-2)(a+2)}{(a+3)(a-3)} + \frac{3}{a-3} = \frac{a^2-4+3}{a^2-9+8} = \frac{a^2-1}{a^2-1} = \frac{a+2}{a+2} \cdot \frac{(a-3)}{(a-3)} = \frac{a-3}{a+2}
\]